CAMBRIDGE ASSESSMENT

STEP Solutions 2010

Mathematics
STEP 9465/9470/9475

October 2010

TEST

STEP Mathematics III 2010: Solutions

Section A: Pure Mathematics

1. The first two parts are obtained by separating off the final term of the summation and expanding the brackets respectively giving $C = \frac{1}{n+1}(nA + x_{n+1})$, and

$$B = \frac{1}{n}\sum_{k=1}^{n} {x_k}^2 - A^2$$

(the latter given in the question) .
By comparison with the expression for *B*,

$$D = \frac{1}{n+1}\sum_{k=1}^{n+1} {x_k}^2 - C^2$$

which by substituting for

$$\frac{1}{n}\sum_{k=1}^{n} {x_k}^2$$

from the expression for B gives

$$D = \frac{1}{n+1}[n(B + A^2) + {x_{n+1}}^2] - C^2$$

.
Substituting for C from the initial result, the required expression can be obtained which can most neatly be written

$$D = \frac{n}{(n+1)^2}[(n+1)B + (A - x_{n+1})^2]$$

Thus $(n+1)D = nB + \frac{n}{n+1}(A - x_{n+1})^2$ yielding the first inequality.

Also, $D - B = \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{1}{n+1}B$ and this quadratic expression is only negative if and only if $(A - x_{n+1})^2 < \frac{n+1}{n}B$.
Rearranging the inequality to make x_{n+1} the subject yields the required result.

2. The expression of cosh *a* in exponentials enables the integral to be written as

$$\int_0^1 \frac{1}{x^2 + x(e^a + e^{-a}) + 1}\,dx$$

which can in turn can be expressed as

$$\int_0^1 \frac{1}{(x + e^a)(x + e^{-a})}\,dx$$

and so employing partial fractions this is

$$\frac{1}{(e^a - e^{-a})}\left[ln\left(\frac{x + e^{-a}}{x + e^a}\right)\right]_0^1$$

The evaluation of this with simplification of logarithms yields

$$\frac{1}{2\sinh a}\left(ln\left(e^a\frac{1+e^a}{1+e^a}\right)\right)$$

giving the required result.

In part (ii), the same technique can be employed for both integrals giving, in the first case

$$\int_1^\infty \frac{1}{(x+e^a)(x-e^{-a})}dx$$
$$=\frac{1}{(e^a+e^{-a})}\left[ln\left(\frac{x-e^{-a}}{x+e^a}\right)\right]_1^\infty$$
$$=\frac{1}{2\cosh a}\left(a+ln\left(\coth\frac{a}{2}\right)\right)$$

and in the second

$$\int_0^\infty \frac{1}{(x^2+e^a)(x^2+e^{-a})}dx$$
$$=\frac{1}{(e^a-e^{-a})}\left[\frac{1}{e^{-\frac{a}{2}}}\tan^{-1}\left(\frac{x}{e^{-\frac{a}{2}}}\right)-\frac{1}{e^{\frac{a}{2}}}\tan^{-1}\left(\frac{x}{e^{\frac{a}{2}}}\right)\right]_0^\infty$$
$$=\frac{1}{2\sinh a}\left(\frac{\pi}{2}2\sinh\frac{a}{2}\right)$$

or alternatively

$$\frac{\pi}{4\cosh\frac{a}{2}}$$

3. The two primitive 4th roots of unity are $\pm i$ so $C_4(x)=(x-i)(x+i)=x^2+1$

$C_1(x)=x-1$, $x^2-1=(x-1)(x+1)$ so $C_2(x)=x+1$,
$x^3-1=(x-1)(x^2+x+1)$ so $C_3(x)=x^2+x+1$
$x^5-1=(x-1)(x^4+x^3+x^2+x+1)$ so $C_5(x)=x^4+x^3+x^2+x+1$
$x^6-1=(x^3-1)(x^3+1)=(x^3-1)(x+1)(x^2-x+1)$ so $C_6(x)=x^2-x+1$

In part (ii), $C_n(x)=0\Rightarrow x^4=-1\Rightarrow x^8=1$ so n is a multiple of 8, and as there are 4 primitive 8th roots of unity, n must be 8.

$x^p=1\Rightarrow x^p-1=0\Rightarrow (x-1)(x^{p-1}+x^{p-2}+x^{p-3}+\cdots+1)$
1 is the only non-primitive root as no power of any other root less than the pth equals unity, because p is prime, so $C_p(x)=x^{p-1}+x^{p-2}+x^{p-3}+\cdots+1$

No root of $C_n(x)=0$ is a root of $C_t(x)=0$ for any $t\neq n$. (For if $t<n$, by the definition of $C_n(x)$, there is no integer t such that $a^t=1$ when $a^n=1$. Similarly, if $t>n$.)
Thus if $C_q(x)\equiv C_r(x)C_s(x)$, and if $C_q(x)=0$, then $C_r(x)=0$ or $C_s(x)=0$, so $q=r$ or $q=s$.

If $q = r$, then $C_q(x) \equiv C_r(x)$, and so $C_s(x) \equiv 1$ which is not possible for positive s, and likewise in the alternative case.

4. (i) As α satisfies both equations, $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + c\alpha + d = 0$, so subtracting these the desired result is simply found.
If $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$, then we may divide by $(a-c)^2$, and find that $-\frac{(b-d)}{(a-c)}$ satisfies $x^2 + ax + b = 0$. But also,
$\left(\frac{(b-d)}{(a-c)}\right)^2 + c\left(-\frac{(b-d)}{(a-c)}\right) + d = \left(\frac{(b-d)}{(a-c)}\right)^2 + a\left(-\frac{(b-d)}{(a-c)}\right) + b + (c-a)\left(-\frac{(b-d)}{(a-c)}\right) + (d-b)$ and so $-\frac{(b-d)}{(a-c)}$ satisfies $x^2 + cx + d = 0$.
On the other hand if there is a common root, then it is found at the start of the question and as it satisfies $\alpha^2 + a\alpha + b = 0$, the required result is found.
If $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$ and $a = c$, then $b = d$ and so the two equations are one and trivially have a common root. Alternatively, if there is a common root and $a = c$, then the initial subtraction yields $b = d$, and so the result is trivially true.

(ii) If $(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0$, then $x^2 + ax + b = 0$ and $x^2 + (q-b)x + r = 0$ have a common root from (i), and so then do $x^2 + ax + b = 0$ and $x(x^2 + ax + b) + x^2 + (q-b)x + r = 0$ which is the required result.
On the other hand, if the two equations have a common root α, then $\alpha^2 + a\alpha + b = 0$ and $\alpha^3 + (a+1)\alpha^2 + q\alpha + r = 0$, and thus so does
$\alpha^3 + (a+1)\alpha^2 + q\alpha + r - \alpha(\alpha^2 + a\alpha + b) = 0$ which is a quadratic equation and we can use the result from (i) again.
Using $= \frac{5}{2}$, $q = \frac{5}{2}$, $r = \frac{1}{2}$, in the given condition, we obtain a cubic equation in b,
$b^3 - \frac{3}{2}b^2 + \frac{1}{4}b + \frac{1}{4} = 0$, which has a solution $b = 1$, meaning the other two can be simply obtained as $b = \frac{1\pm\sqrt{5}}{4}$.

5. The line CP can be shown to have equation $(1-n)y = x - an$ and so R is $\left(0, \frac{an}{n-1}\right)$
So, similarly, S must be $\left(\frac{am}{m-1}, 0\right)$.
Thus RS has equation $n(m-1)x + m(n-1)y = amn$ and PQ has equation $mx + ny = amn$.
As the coordinates of T satisfy both equations, they satisfy their difference which is
$(mn - n - m)(x + y) = 0$. As RS and PQ intersect, $\frac{n}{m} \neq \frac{m(n-1)}{n(m-1)}$ which yields
$(m-n)(mn - m - n) \neq 0$ and hence $(mn - m - n) \neq 0$ implying that T's coordinates satisfy $x + y = 0$ giving the desired result. (Alternatively, $mn - m - n = 0 \Leftrightarrow n = \frac{m}{m-1} < 0$, which is a contradiction.)
The construction can be achieved more than one way, but one is to label the given square ABCD anti-clockwise, choose points on AB and AD different distances from A, label them P and Q, construct CP and CQ, and find their intersections with AD and AB, R and S, respectively, and find the intersection of PQ and RS, label it T, then TA is perpendicular to AC. Rotating the labelling through a right angle and repeating three more times achieves the desired square.

6. P_1 is $(\cos\varphi, \sin\varphi, 0)$, P_2 is $(\cos\varphi\cos\lambda, \sin\varphi\cos\lambda, \sin\lambda)$, Q_1 is $(-\sin\varphi, \cos\varphi, 0)$, Q_2 is $(-\sin\varphi, \cos\varphi, 0)$, R_1 is $(0,0,1)$ and R_2 is $(-\cos\varphi\sin\lambda, -\sin\varphi\sin\lambda, \cos\lambda)$.
The scalar product $OP_2 \cdot OP_0$ gives the quoted result immediately. The direction of the axis can be found from the vector product $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}\cos\varphi\cos\lambda\\ \sin\varphi\cos\lambda\\ \sin\lambda\end{pmatrix}$ giving the direction of the axis as $\begin{pmatrix}0\\ -\sin\lambda\\ \sin\varphi\cos\lambda\end{pmatrix}$.

7. The initial result can be obtained by differentiating y directly twice obtaining
$\frac{dy}{dx} = -\sin(m\sin^{-1}x)\frac{m}{\sqrt{1-x^2}}$
$\frac{d^2y}{dx^2} = -\cos(m\sin^{-1}x)\frac{m^2}{1-x^2} - \sin(m\sin^{-1}x)\frac{mx}{(1-x^2)^{\frac{3}{2}}}$ and substituting into the LHS.
(Slightly more elegant is to rearrange as $\cos^{-1}y = m\sin^{-1}x$, differentiate and then square to obtain $(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$ and then differentiate a second time.)
The two similar results are $(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} + (m^2-1)\frac{dy}{dx} = 0$ and
$(1-x^2)\frac{d^4y}{dx^4} - 5x\frac{d^3y}{dx^3} + (m^2-4)\frac{d^2y}{dx^2} = 0$, which lead to the conjecture
$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + (m^2-n^2)\frac{d^ny}{dx^n} = 0$ which is proved simply by induction.

Using $= 0$, we find that $y = 1$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -m^2$, $\frac{d^3y}{dx^3} = 0$, $\frac{d^4y}{dx^4} = m^2(m^2-4)$
and so the Maclaurin series commences $y = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-2^2)}{4!}x^4 + \cdots$

Now replacing x by $\sin\theta$,
$\cos m\theta = 1 - \frac{m^2}{2!}x^2 + \frac{m^2(m^2-2^2)}{4!}x^4 + \cdots = 1 - \frac{m^2}{2!}\sin^2\theta + \frac{m^2(m^2-2^2)}{4!}\sin^4\theta + \cdots$
All the odd differentials are zero, and the even ones are $(-1)^{k+1}m^2(m^2-2^2)\ldots(m^2-(2k)^2)$, so if m is even all the terms are zero from a certain point (when $m = 2k$) and thus the series terminates and is a polynomial in $\sin\theta$, of degree m.

8. Substituting for $P(x)$, the desired integral is seen to be the reverse of the quotient rule, i.e.

$$\frac{R(x)}{Q(x)}(+k)$$

To choose a suitable function $R(x)$ in part (i), substitution of $R(x) = a + bx + cx^2$ and $Q(x) = 1 + 2x + 3x^2$ in the given expression yields a quadratic equation, and equating the coefficients of the powers of x gives $5 = -3b + 2c$, $-2 = -3a + c$, $-3 = -2a + b$.
These three equations are linearly dependent and so their solution is not unique.
Choosing, for example $a = 0$, $= -3$, $c = -2$ and then $a = 1$, $b = -1$, $c = 1$ gives solutions which are related by $\frac{1-x+x^2}{1+2x+3x^2} = \frac{1+2x+3x^2-3x-2x^2}{1+2x+3x^2} = 1 + \frac{-3x-2x^2}{1+2x+3x^2}$ i.e. the same bar the

arbitrary constant.

(ii) Rearranging the equation to be solved as $\frac{dy}{dx}+\frac{(\sin x-2\cos x)}{(1+\cos x+2\sin x)}y=\frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)}$, the integrating factor is $e^{\int\frac{(\sin x-2\cos x)}{(1+\cos x+2\sin x)}dx}=e^{-\ln(1+\cos x+2\sin x)}=\frac{1}{1+\cos x+2\sin x}$

As a result, the RHS we require to integrate is $\frac{(5-3\cos x+4\sin x)}{(1+\cos x+2\sin x)^2}$

Repeating similar working to part (i), except with $Q(x)=1+\cos x+2\sin x$ and $R(x)=a+b\sin x+c\cos x$, gives three linearly dependent equations,

$5=b-2c$, $-3=b-2a$, $4=a-c$

Choosing e.g. $=4$, $b=5$, $c=0$, the solution is $y=4+5\sin x+k(1+\cos x+2\sin x)$

Section B: Mechanics

9. Resolving radially inwards for the mass P, $mg\sin\theta-R=\frac{mv^2}{a}$,
where R is the normal reaction of the block on P, and v is the (common) speed of the masses when OP makes an angle θ with the table.

Conserving energy, $\frac{1}{2}mv^2+\frac{1}{2}Mv^2+mga\sin\theta-Mga\theta=0$, and making v^2 the subject of this formula to substitute in the first equation re-arranged for R,

$R=mg\sin\theta-\frac{2mg(M\theta-m\sin\theta)}{m+M}=\frac{mg\big((3m+M)\sin\theta-2M\theta\big)}{m+M}$ is found.

Remaining in contact requires this expression to be non-negative for all , $0\leq\theta\leq\frac{\pi}{2}$.

Considering the graphs of $y=a\sin\theta$ and $y=b\theta$ for $0\leq\theta\leq\frac{\pi}{2}$,

$a\sin\theta-b\theta\geq 0,\forall\theta,0\leq\theta\leq\frac{\pi}{2}$ if and only if $a\sin\theta-b\theta\geq 0$ for $\theta=\frac{\pi}{2}$

so $R\geq 0$ for all $\theta,0\leq\theta\leq\frac{\pi}{2}$ if and only if $(3m+M)\sin\frac{\pi}{2}-2M\frac{\pi}{2}\geq 0$ which gives the required result.

10. Resolving perpendicularly to OB, $ma\ddot{\phi}=-T\cos\left(\frac{\pi}{2}-\theta-\phi\right)$, where the tension in the elastic string is $T=\lambda\frac{PB-c}{c}$. The sine rule $\frac{a}{\sin\theta}=\frac{PB}{\sin\phi}$

Putting these three results together gives the required expression.

Also from the sine rule, $\frac{b}{\sin(\theta+\phi)}=\frac{a}{\sin\theta}$, so for ϕ and θ small, $\frac{b}{\theta+\phi}\approx\frac{a}{\theta}$ yielding the desired result.

From this result, θ may be made the subject of the formula, so that the result

$ma\ddot{\phi}=-\lambda\left(\frac{a\sin\phi}{c\sin\theta}-1\right)\sin(\theta+\phi)$, which for small angles becomes

$ma\ddot{\phi}\approx-\lambda\left(\frac{a\phi}{c\theta}-1\right)(\theta+\phi)$ can be written $\ddot{\phi}\approx-\frac{\lambda}{ma}\left(\frac{b-a-c}{c}\right)\left(\frac{b}{b-a}\right)\phi$

and hence the period is $\tau\approx 2\pi\sqrt{\frac{mac(b-a)}{\lambda b(b-a-c)}}$.

11. If the acceleration of the block is a', and the acceleration of the bullet is a'', then $R - \mu(M+m)g = Ma'$ and $-R = ma''$,
so the relative acceleration $a = a' - a'' = \frac{R}{m} + \frac{R-\mu(M+m)g}{M}$

The initial velocity of the bullet relative to the block is $-u$ and the final velocity of the bullet relative to the block is 0. If the time between the bullet entering the block and stopping moving through the block is T, then using" $v = u + at$ ", $0 = -u + \left(\frac{R}{m} + \frac{R-\mu(M+m)g}{M}\right)T$
For the block, the initial velocity is 0 , the final velocity is v , and again using $v = u + at$,

$v = a'T = \frac{R-\mu(M+m)g}{M} \frac{u}{\left(\frac{R}{m}+\frac{R-\mu(M+m)g}{M}\right)}$ and so
$av = \left(\frac{R}{m} + \frac{R-\mu(M+m)g}{M}\right) \frac{R-\mu(M+m)g}{M} \frac{u}{\left(\frac{R}{m}+\frac{R-\mu(M+m)g}{M}\right)} = \frac{Ru-\mu(M+m)gu}{M}$ as required.
If the distance moved by the block whilst the bullet is moving through the block is s,
using" $v^2 = u^2 + 2as$ ", $v^2 = 2a's$ andso $s = \frac{v^2}{2a'} = \frac{Mv^2}{2(R-\mu(M+m)g)} = \frac{Mv^2}{2\frac{Mav}{u}} = \frac{uv}{2a}$

Once the bullet stops moving through the block, the next initial velocity of block/bullet is v, the final velocity is 0, the acceleration is $-\mu g$, so the distance moved s'using
"$v^2 = u^2 + 2as$" is given by $0 = v^2 - 2\mu g s'$ i.e. $s' = \frac{v^2}{2\mu g}$
Thus the total distance moved is $\frac{uv}{2a} + \frac{v^2}{2\mu g} = \frac{v}{2\mu g a}[\mu g u + av]$
$= \frac{v}{2\mu g a}\left[\mu g u + \frac{Ru-\mu(M+m)gu}{M}\right]$

$= \frac{uv}{2\mu g}\left[\frac{R-\mu mg}{Ma}\right]$

$= \frac{uv}{2\mu g}\left[\frac{R-\mu mg}{M}\right] \frac{1}{\frac{R}{m}+\frac{R-\mu(M+m)g}{M}}$

$= \frac{uv}{2\mu g}\left[\frac{R-\mu mg}{M}\right] \frac{Mm}{(M+m)(R-\mu mg)} = \frac{muv}{2(M+m)\mu g}$

If $R < (M+m)\mu g$, then the block does not move, and the bullet penetrates to a depth $\frac{mu^2}{2R}$.

Section C: Probability and Statistics

12. $S - rS = 1 + dr + dr^2 + \cdots + dr^n + \cdots$ which is 1 plus an infinite GP. Summing that GP and making S the subject produces the displayed result.

$E(A) = 1a + 2(1-a)a + 3(1-a)^2a + \cdots + n(1-a)^{n-1}a + \cdots$ so making use of the first result with $d = 1, r = (1-a)$, $E(A) = a\left\{\frac{1}{1-(1-a)} + \frac{(1-a)}{(1-(1-a))^2}\right\} = a\left\{\frac{1}{a} + \frac{1-a}{a^2}\right\} = \frac{1}{a}$

$\alpha = a + (1-a)(1-b)\alpha = a + a'b'\alpha$ or alternatively, $\alpha = a + a'b'a + a'^2b'^2a + \cdots$ which both lead to the required result.

$\beta = 1 - \alpha = \frac{a'b}{1-a'b'}$ or alternatively, $\beta = a'b + a'^2b'b + a'^3b'^2b + \cdots = \frac{a'b}{1-a'b'}$

The expected number of shots, S, is given by

$E(S) = 1a + 2a'b + 3a'b'a + 4a'^2b'b + 5a'^2b'^2a + \cdots$

$= a\{1 + 3a'b' + 5a'^2b'^2 + \cdots\} + 2a'b\{1 + 2a'b' + \cdots\}$

which using the initial result of the question $= a\left[\frac{1}{1-a'b'} + \frac{2a'b'}{(1-a'b')^2}\right] + 2a'b\left[\frac{1}{1-a'b'} + \frac{a'b'}{(1-a'b')^2}\right]$ and can be shown to simplify to the required expression.

13. $Corr(Z_1, Z_2) = 0$

$E(Y_2) = E\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right) = \rho_{12}E(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}E(Z_2) = 0$

$Var(Y_2) = Var\left(\rho_{12}Z_1 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_2\right) = \rho_{12}^2 Var(Z_1) + (1-\rho_{12}^2)Var(Z_2)$

$= \rho_{12}^2 + (1-\rho_{12}^2) = 1$

As $E(Y_1) = E(Y_2) = 0$ and $Var(Y_1) = Var(Y_2) = 1$,

$Corr(Y_1, Y_2) = \frac{Cov(Y_1,Y_2)}{\sqrt{Var(Y_1)Var(Y_2)}} = Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$

$= E\left(\rho_{12}Z_1^2 + (1-\rho_{12}^2)^{\frac{1}{2}}Z_1Z_2\right) = \rho_{12}Var(Z_1) + (1-\rho_{12}^2)^{\frac{1}{2}}E(Z_1)E(Z_2) = \rho_{12}$

$E(Y_3) = E(aZ_1 + bZ_2 + cZ_3) = aE(Z_1) + bE(Z_2) + cE(Z_3) = 0$ is given.

$Var(Y_3) = 1$ implies $a^2 + b^2 + c^2 = 1$

$Corr(Y_1, Y_3) = \rho_{13}$ implies $a = \rho_{13}$ as seen before.

$Corr(Y_2, Y_3) = \rho_{23}$ implies $\rho_{12}a + (1-\rho_{12}^2)^{\frac{1}{2}}b = \rho_{23}$

and hence $a = \rho_{13}$, $b = \frac{\rho_{23} - \rho_{12}\rho_{13}}{(1-\rho_{12}^2)^{\frac{1}{2}}}$, $c = \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{(1-\rho_{12}^2)}}$

$X_i = \mu_i + \sigma_i Y_i$ for $i = 1,2,3$ as $E(X_i) = E(\mu_i + \sigma_i Y_i) = E(\mu_i) + E(\sigma_i Y_i) = \mu_i + \sigma_i E(Y_i) = \mu_i$,

$Var(X_i) = Var(\mu_i + \sigma_i Y_i) = Var(\sigma_i Y_i) = \sigma_i^2 Var(Y_i) = \sigma_i^2$, and

$Corr(X_i, X_j) = Corr(Y_i, Y_j) = \rho_{ij}$ as a linear transformation does not affect correlation.